case, D contains the eigenvalues in the diagonal. G then is an orthonormal modal matrix of D.

$$G = \begin{bmatrix} g_{11} & g_{21} \cdots g_{n1} \\ g_{12} & g_{22} \cdots g_{n2} \\ \vdots & \\ g_{1n} & g_{2n} \cdots g_{nn} \end{bmatrix} \quad (13)$$

and

$$D = \begin{bmatrix} d_{11} & 0 & \ldots 0 \\ 0 & d_{22} & \ldots 0 \\ \vdots & & \\ 0 & 0 & \ldots d_{nn} \end{bmatrix} = GDG^T \quad (14)$$

where the jth column in G is the eigenvector corresponding to the jth eigenvalue $\mathbf{d}_{jj}$. This method is easy to use since computer programs for finding the eigenvalues and eigenvectors of a real symmetric matrix are easily available.[5]

The second method is to look for a matrix G which is triangular; i.e.,

$$G = \begin{bmatrix} g_{11} & g_{12} \cdots g_{1n} \\ 0 & g_{22} \cdots g_{2n} \\ \vdots & \\ 0 & 0 \quad \ldots g_{nn} \end{bmatrix} \quad (15)$$

where

$$g_{ii}^2 = A_i/A_{i-1} \quad (16)$$

and A_{i-k} is the determinant obtained by striking out the last k rows and last k columns from the matrix D. The remaining g_{ij} are obtained from

$$d_{jk} = \Sigma g_{ij} g_{ik}, \quad k = j + 1, \ldots, n \quad (17)$$

the procedure outlined by Graham appears to be equivalent to this latter method of diagonalization.

Two-Constraint Minimization Problem

It may now be expected that the drag minimization calculation would be simplified. Consider for example, the problem of obtaining minimum drag subject to given lift and pitching moment constraints. The problem is

$$\text{Minimize} \quad C_D = \Sigma \epsilon_i^2 d_{ii}$$

$$\text{subject to} \quad \Sigma \epsilon_i l_i = C_L \quad (18)$$

$$\Sigma \epsilon_i m_i = C_M$$

where d_{ii}, l_i, *and* m_i are the drag, lift and moment coefficients, respectively, due to unit intensity of the ith member in an orthognal set. C_L and C_M are prescribed lift and moment coefficients, respectively. The corresponding unconstrained problem is

$$\text{Minimize} \quad \Sigma \epsilon_i^2 d_{ii} + \lambda_1(\Sigma \epsilon_i l_i - C_L) + \lambda_2(\Sigma \epsilon_i m_i - C_M) \quad (19)$$

where λ_1 and λ_2 are the Lagrange multipliers. The necessary conditions that Eq. (19) be a minimum are

$$2\epsilon_i d_{ii} + \lambda_1 l_i + \lambda_2 m_i = 0; \; i = 1, 2, \ldots, n \quad (20)$$

$$\Sigma \epsilon_i l_i = C_L$$
$$\Sigma \epsilon_i m_i = C_M \quad (21)$$

Multiplying Eq. (20) by ϵ_i and summing over i and using the two constraint equations (21), we have

$$\lambda_1 = -(\lambda_2 C_M + 2C_D)/C_L \quad (22)$$

Substituting for λ_1 in Eq. (20) we have

$$\epsilon_i = \frac{1}{2d_{ii}} \left[\frac{2C_D l_i}{C_L} - \lambda_2 \left(m_i - \frac{C_M l_i}{C_L} \right) \right] \quad (23)$$

and for C_D we find

$$C_D = \frac{C_M^2 \Sigma \frac{l_j^2}{d_{jj}} - 2C_M C_L \Sigma \frac{l_j m_j}{d_{jj}} + C_L^2 \Sigma \frac{m_j^2}{d_{jj}}}{\Sigma \frac{l_j^2}{d_{jj}} \Sigma \frac{m_j^2}{d_{jj}} - \left(\Sigma \frac{l_j m_j}{d_{jj}} \right)^2} \quad (24)$$

Equation (20) then yields

$$\lambda_2 = \left(2C_M - \frac{2C_D}{C_L} \Sigma \frac{l_j m_j}{d_{jj}} \right) \Big/ \left(\frac{C_M}{C_L} \Sigma \frac{l_j m_j}{d_{jj}} - \Sigma \frac{m_j^2}{d_{jj}} \right) \quad (25)$$

When only the lift constraint is given, setting $\lambda_2 = 0$ gives the results obtained in Ref. 1.

The Langrange multiplier method when applied to nonorthogonal loads may give rise to an ill-conditioned matrix and higher precision may be required to obtain reasonable results. The orthogonal loading method alleviates this problem and should be suitable for digital calculations.

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Frequency Determination from Similarity Considerations

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Nomenclature

$\bar{\alpha}_i, \bar{\alpha}_i', \bar{\alpha}_i''$	= bending frequency coefficients
$\bar{\beta}_i, \bar{\beta}_i', \bar{\beta}_i''$	= torsional frequency coefficients
ρ	= fluid density
ρ_s	= structural density
σ_b	= allowable bending stress
σ_s	= allowable shear stress
μ	= aircraft relative mass
ω	= frequency (rad/sec)
a'	= sonic velocity
$\bar{c}$	= mean aerodynamic chord
c_1', c_2', c_1'', c_2''	= frequency coefficients
g	= gravitational acceleration
k_i	= reduced frequency parameter
l	= semispan length
m	= wing mass per unit length
t	= wing thickness at root
E	= modulus of elasticity

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G	= shear modulus
I	= area moment of inertia
I_0	= wing mass moment of inertia per unit span
J	= polar moment of inertia
M	= aircraft gross mass
$\bar{M}$	= Mach number
W	= aircraft gross takeoff weight

Subscripts and Superscripts

b	= bending mode
i	= mode designation
t	= torsional mode

I. Introduction

AN important quantity which appears in the analysis for dynamic response of an aircraft when wing bending and torsional flexibility effects are included, is the ratio of the respective natural frequencies to the forcing frequency. For preliminary analysis, it would be advantageous to be able to determine such values in an approximate way by considering only a few of the gross characteristics of the aircraft. The application of similarity to this task seems appropriate. Various aircraft similarity parameters may be investigated as to their utilization in evaluating the required wing frequencies. Those of importance will now be determined.

II. Natural Frequency

If one considers a cantilevered uniform beam to approximate, in a simple case, a wing, the exact solution for the bending natural frequency is given as

$$\omega_i^b = \alpha_i(EI/ml^4)^{1/2} \tag{1}$$

where for the fundamental mode $\alpha_1 = 3.515$.

The quantities under the radical may be expressed in terms of aircraft parameters in the following manner:

$$I = c_1'(Wlt/\sigma_b) \tag{2}$$

$$m = c_2'(\rho_s t\bar{c}) \tag{3}$$

$$\mu = (M/\rho\bar{c}^2 l) \tag{4}$$

Equations (2, 3, and 4) may be substituted into Eq. (1) to yield

$$\omega_i^b = \bar{\alpha}_i(\{E/\sigma_b\}\{\bar{c}/l\}\{\rho/\rho_s\}\{g/l\}\ \mu)^{1/2} \tag{5}$$

where

$$\bar{\alpha}_i = \alpha_i(c_1'/c_2')^{1/2}$$

If the type of craft under consideration is similar geometrically and materially to other known craft, then these may be used to evaluate the coefficient trends. It may also be shown that the nondimensional quantities (E/σ_b), $(\bar{c}/l)$, and (ρ/ρ_s) will be constant.[1] For example, if one considers subsonic transport jet aircraft, both domestic and foreign, $(\bar{c}/l)$ is found to be approximately 0.4 for all craft regardless of size. Material properties obviously indicate that (E/σ_b) and (ρ/ρ_s) are also constant. It is, therefore, possible to express the frequency as

$$\omega_i^b = \bar{\alpha}_i'(\mu g/l)^{1/2} \tag{6}$$

where $\bar{\alpha}_i'$ is equal to

$$\bar{\alpha}_i' = \alpha_i[\{c_1'/c_2'\}\{E/\sigma_b\}\{\bar{c}/l\}\{\rho/\rho_s\}]^{1/2} \tag{7}$$

The frequency relation, Eq. (6) may be seen to be similar to that for a simple pendulum weighted by the aircraft relative mass μ and the coefficient $\bar{\alpha}_i'$. The relative mass is plotted against gross takeoff weight in Fig. 1 for subsonic jets and turboprop craft traveling at thirty thousand feet.

The torsional frequency of a cantilevered wing may also be related to analogous nondimensional parameters. The torsional frequency is usually expressed as

$$\omega_i^t = \beta_i(GJ/I_o l^2)^{1/2} \tag{8}$$

but the following substitutions may be made:

$$J = c_1''(W\bar{c}^2/\sigma_s) \tag{9}$$

$$I_0 = c_2''(\rho_s\bar{c}^3 t) \tag{10}$$

Torsional frequency then takes the form

$$\omega_i^t = \bar{\beta}_i'(\mu g/l)^{1/2} \tag{11}$$

which again is an analogous expression to the simple pendulum weighted by the coefficient $\bar{\beta}_i'$ and the craft's relative mass μ. In Eq. (11)

$$\bar{\beta}_i' = \beta_i(\{G/\sigma_s\}\{\rho/\rho_s\}\{\bar{c}/l\}\{\bar{c}_1''/\bar{c}_2''\})^{1/2} \tag{12}$$

The nondimensional quantities (G/σ_s), (E/σ_b), (ρ/ρ_s), $(\bar{c}/l)$, (c_1'/c_2'), and (c_1''/c_2'') may be established once and for all. Although (c_1'/c_2') and (c_1''/c_2'') are not constants, a plot may be established against relative mass from known characteristics of similar craft. Therefore, for any geometrically and materially similar craft, it is only necessary to establish the relative mass from Fig. 1 and specify the semispan length in order that the frequencies be known in an approximate way.

The choice of parameters to be used is not unique. Other possible schemes may be derived. It is found that several terms in the preceding relations are dependent on the flight altitude. Such quantities are the relative mass μ and the density ρ.

III. Reduced Frequency Parameter

The reduced frequency parameter may be expressed as a function of Mach number and sonic velocity

$$k_i = \omega_i\bar{c}/2u_0 \tag{13}$$

The bending reduced frequency may be expressed using Eq. (6)

$$k_i^b = \bar{\alpha}_i''\left(\frac{1}{a'\bar{M}}\right)(\bar{c}g\mu)^{1/2} \tag{14}$$

where

$$\bar{\alpha}_i'' = \bar{\alpha}_i'/2(\bar{c}/l)^{1/2}$$

Similarly, the relation for the torsional frequency is given by substitution of Eq. (11) into Eq. (13).

$$k_i^t = \bar{\beta}_i''\left(\frac{1}{a'\bar{M}}\right)(\bar{c}g\mu)^{1/2} \tag{15}$$

where $\quad \bar{\beta}_i'' = \beta_i'\mu(\bar{c}/l)^{1/2}$

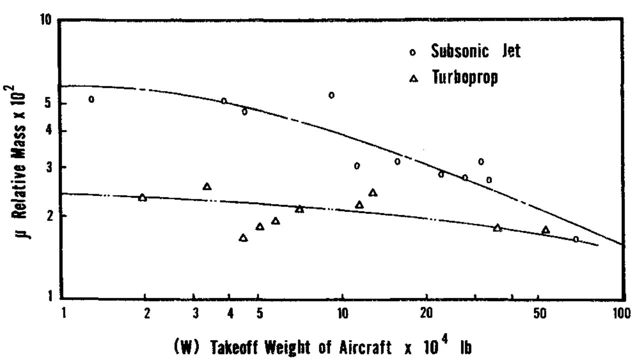

Fig. 1 Takeoff weight vs relative mass.

The reference altitude upon which μ, ρ, and a' are to be selected is clearly arbitrary.

IV. Conclusions

For a specific type of craft which is structurally and geometrically similar, a plot may be established for the coefficients $\bar{\alpha}_i'$ and $\bar{\beta}_i'$ against relative mass μ. For engineering approximations of the wing natural frequencies, it is only necessary to compute μ, interpolate values of the coefficients $\bar{\alpha}_i'$ and $\bar{\beta}_i'$, and specify the semispan length. This method of approximation is feasible for all types of craft as long as similarity characteristics within each class is preserved.

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Cambered Joukowsky Airfoil in a Nonuniform Weak Shear Flow

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THE analysis by Tsien,[1] Sowryda,[2] and Jones[3] of symmetrical and cambered Joukowsky airfoils placed in two-dimensional uniform and nonuniform shear flows have predicted an increase in lift and pitching moment characteristics compared to the values of uniform flow. The results for symmetrical Joukowsky airfoils have been tested experimentally by Vidal[4]; Vidal, Hilton, and Curtis[5]; and Ludwig and Erickson Jr.[6] for the values of shear parameter equal to about two to five. Furthermore, their experiments carried out in a two dimensional nonuniform shear flow simulating a propeller slipstream showed that the airfoil characteristics depended upon the location of the airfoil and the product of local stream shear with the shear derivative.

An experimental investigation was undertaken to extend the measurements to the cambered Joukowsky airfoil placed in a two-dimensional nonuniform weak shear flow. The results of this investigation are presented in this Note.

Experiments

A 6 in. chord, 12 in. span Joukowsky airfoil of thickness ratio τ of 0.15 and camber of 0.10 was made out of seasoned wood. The coordinates of the Joukowsky profile were obtained by the method given in Ref. 7. Twenty-three static pressure holes of diameter 0.032 in. were made along the midspan chord, and an equal number of pressure leads were taken out of a ¾ in. o.d. tube fitted in the center of the left tip of the airfoil. The axis of the ¾ in. tube coincided with the midchord axis, and the airfoil was pitched about this axis. To obtain the nonuniform weak shear flow, a screen using horizontal aluminum rods of ¼ in. and ½ in. diam was built following the steps out-

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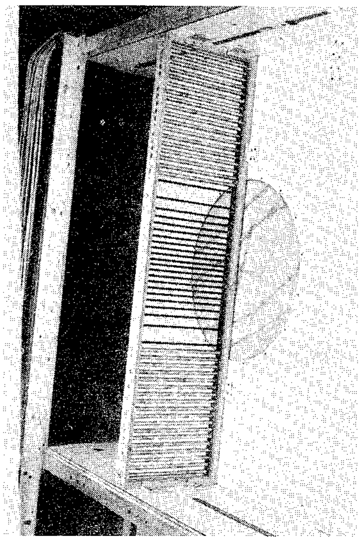

Fig. 1a Screen; flow from left to right.

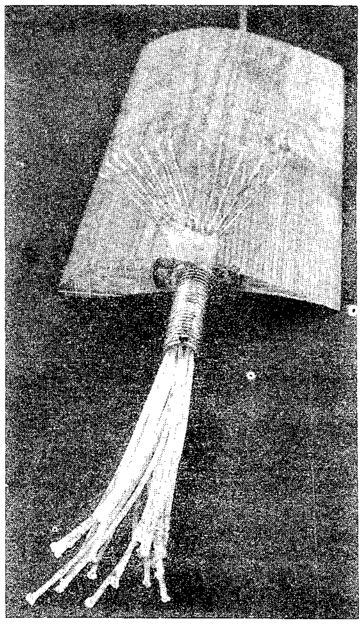

Fig. 1b Airfoil.

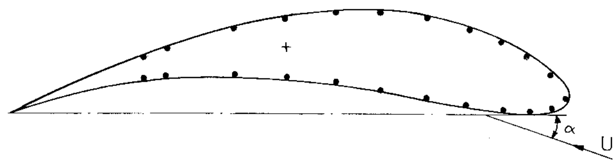

Fig. 1c Cambered Joukowsky airfoil: chord 15 cms: thickness 15%; camber 10%; static pressure taps (·).

lined in Ref. 5 and 6. Pictures of the screen mounted in the wind-tunnel test section, of the cambered Joukowsky airfoil, and a sketch of the contour of the airfoil chord with static pressure locations marked by dots are shown in Figs. 1a–c, respectively.

The experiments were carried out in a closed-circuit, closed-jet, low-speed wind tunnel at IIT Kanpur. The test section of the wind tunnel is 4 ft high, 1 ft wide, and 5.5 ft long. It has a contraction ratio of 9.5 and a turbulence level of 1.4% at the maximum wind speed of 180 fps. Two fans mounted one on top of the other in the return circuit are driven by two motors of 15 hp each. A velocity traverse in the vertical direction made in the clear test section showed a ±1.5% variation in mean velocity across the test section height.

The two-dimensional airfoil which spanned the 1 ft width of the test section was located 36 in. downstream of